数理解析研究所講究録 1370 巻 2004 年 102-116

Extension of Bing Maps

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Abstract

In [7], M. Levin proved that the set of all Bing maps of a compact metric space to the unit interval constitutes a G_δ-dense subset of the space of maps. In [6], J. Krasinkiewicz independently proved that the set of all Bing maps of a compact metric space to an n-dimensional manifold ($n \geq 1$) constitutes a G_δ-dense subset of the space of maps. In [9], J. Song and E. D. Tymchatyn solved some problems of J. Krasinkiewicz [6]: They proved that the set of all Bing maps of a compact metric space to a nondegenerate connected polyhedron (or a 1-dimensional locally connected continuum) constitutes a G_δ-dense subset of the space of maps. In this note, by using methods of Levin [7] and Krasinkiewicz [6], we prove the extension theorem of Bing maps which is slightly precise than the theorem of J. Song and E. D. Tymchatyn.

1 Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous functions. We denote the unit interval $[0,1]$ by $\mathbb{I}$. An *arc* is a space which is homeomorphic to $\mathbb{I}$. If X is a compact metrizable space and Y is a space, $C(X, Y)$ denotes the space of all continuous maps from X to Y endowed with sup metric. A compact metrizable space is called a *compactum*, and a *continuum* means a connected compactum. A map f is called a *ε-map* if all diameters of fibers of f are smaller than ε. A continuum is said to be *indecomposable* if it is not sum of two proper subcontinua. A compactum is called a *Bing compactum* (or said to be *hereditarily indecomposable*) if each of its subcontinua is indecomposable. A map is called a *Bing map* if each of its fibers is a Bing compactum. In [7], M. Levin proved the following theorem.

Theorem 1 (M. Levin [7]) *For each compactum X, the set of all Bing maps in $C(X,\mathbb{I})$ is a G_δ-dense subset in $C(X,\mathbb{I})$.*

On the other hand, J. Krasinkiewicz proved the next theorem independently.

Theorem 2 (J. Krasinkiewicz [6])*Let X be a compactum and let Y be an n-dimensional manifold ($n \geq 1$). Then the set of all Bing maps in $C(X,Y)$ is a G_δ-dense subset in $C(X,Y)$.*

Note that Theorem 2 is a generalization of Theorem 1. In [6], J. Krasinkiewicz poses the following problem: If Y in Theorem 2 is an other space (for example, dendrite, dendroid, polyhedron, locally connected continuum, the Menger universal curve, AR, ANR), does Theorem 2 hold? In [9], J. Song and E. D. Tymchatyn solved the problems of J. Krasinkiewicz: In particular, they proved the following:

Theorem 3 (J. Song and E. D. Tymchatyn [9]) *The set of all Bing maps of a compact metric space to a nondegenerate connected polyhedron (or a 1-dimensional locally connected continuum) constitutes a G_δ-dense subset of the space of maps.*

In this note, we prove the following theorem by using methods of Levin [7] and Krasinkiewicz [6], which is more precise than the above theorem of J. Song and E. D. Tymchatyn. The proofs are somewhat different from one of J. Song and E. D. Tymchatyn [9]. For case of graphs, we use an idea of M. Levin [7], and for general case of polyhedra, we will use an idea of J. Krasinkiewicz [6].

Theorem 4 (Extension Theorem of Bing Maps) *Let X be a compactum and let A be a closed subset in X. Let $\mathcal{K}$ be a finite simplicial complex such that $|\mathcal{K}|$ is a nondegenerate connected polyhedron and let $\mathcal{L}$ be a subcomplex of $\mathcal{K}$. If $f : A \to |\mathcal{L}|$ is a Bing map and $\tilde{f} : X \to |\mathcal{K}|$ is a map with $\tilde{f}|A = f$ and $\tilde{f}^{-1}(|\mathcal{L}|) = A$, then for any $\varepsilon > 0$ there exists a Bing map $g : X \to |\mathcal{K}|$ such that $g|A = f$ and $d(\tilde{f}, g) < \varepsilon$.*

As a corollary, we obtain the theorem of J. Song and E. D. Tymchatyn. Also, we investigate surjective Bing maps from continua to polyhedra.

2 Preliminaries

In this section, first we give some definitions which are used in this paper.

Notation 5 Let X be a space and let d be a metric on X. We denote the identity map of X by id_X. For a subset $A \subset X$ and $\delta > 0$, denote $B(A,\delta) = \{x \in X \mid \text{there exists } a \in A \text{ such that } d(x,a) < \delta\}$, $\operatorname{diam} A = \sup\{d(x,y) |\ x,y \in A\}$, $\operatorname{cl} A = \{x \in X |$ if U is a neighborhood of x then $U \cap A \neq \emptyset\}$, $\operatorname{int} A = \{x \in X |$ there exists a neighborhood V of x such that $V \subset A\}$. If $\mathcal{A}$ is a family of subsets of X, denote $\operatorname{mesh}\mathcal{A} = \sup\{\operatorname{diam} A | A \in \mathcal{A}\}$. If σ is a simplex, we denote the boundary of σ by $\partial\sigma$. If $\mathcal{K}$ is a simplicial complex and $n \in \mathbb{N}$, we denote $\mathcal{K}^{(n)} = \{\sigma \in \mathcal{K} | \dim \sigma \leq n\}$ and $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$. For each arc I and $x \neq y \in I$, $[x,y]_I$ means an arc in I from x to y and $[x,y)_I$, $(x,y]_I$, $(x,y)_I$ mean $[x,y]_I \setminus \{y\}$, $[x,y]_I \setminus \{x\}$, $[x,y]_I \setminus \{x,y\}$ respectively.

Now we will give the definition of *D-crooked*. The definition of D-crooked was originally introduced in [2], and the definition below was given in [7].

Definition 6 Let $\mathcal{D} = \{(F_0, F_1, V_0, V_1) \mid F_0, F_1$ are disjoint closed subsets in $\mathbb{I}^{\aleph_0}$ and V_0, V_1 are disjoint open neighborhoods of F_0, F_1 in $\mathbb{I}^{\aleph_0}\}$ and $D = (F_0, F_1, V_0, V_1) \in \mathcal{D}$. A subspace $X \subset \mathbb{I}^{\aleph_0}$ is *D-crooked* if there exists an open neighborhood U of X in $\mathbb{I}^{\aleph_0}$ such that for any map $f : \mathbb{I} \to U$ with the property $f(0) \in F_0$ and $f(1) \in F_1$, there exist t_0, t_1 with $0 < t_0 < t_1 < 1$ such that $f(t_0) \in V_1$ and $f(t_1) \in V_0$. A map is said to be *D-crooked* if each of its fibers is D-crooked.

Clearly, subspaces of D-crooked spaces are also D-crooked. M. Levin obtained the following propositions in [7].

Proposition 7 (M. Levin [7]) *If $A \subset \mathbb{I}^{\aleph_0}$ is D-crooked, then there exists a neighborhood $U \subset \mathbb{I}^{\aleph_0}$ of A such that U is D-crooked.*

Proposition 8 (M. Levin [7]) *A compactum $A \subset \mathbb{I}^{\aleph_0}$ is a Bing compactum if and only if A is D-crooked for each $D \in \mathcal{D}$.*

Proposition 9 (M. Levin [7]) *There exist $D_1, D_2, \ldots \in \mathcal{D}$ such that for any compactum $A \subset \mathbb{I}^{\aleph_0}$, A is a Bing compactum if and only if A is D_i-crooked for each $i = 1, 2, \ldots$*

The next theorem was proved by R. H. Bing. Many authors used the theorem to reach important conclusions (for example, the theorem is used in the proof of Theorem 1).

Theorem 10 (R. H. Bing [2])*Let X be a compactum and let A, B be disjoint closed subsets in X. Then there exists a Bing compactum L such that L is a partition between A and B.*

Now, we recall the definition of inverse limits. Let $\{X_i, f_i\}_{i=1}^{\infty}$ be a double sequence of spaces X_i, called *coodinate spaces*, and maps $f_i : X_{i+1} \to X_i$, called *bonding maps*. Then *inverse limit* of $\{X_i, f_i\}_{i=1}^{\infty}$, denoted by $\varprojlim\{X_i, f_i\}$, is the subspace of $\prod_{i=1}^{\infty} X_i$ defined by $\varprojlim\ \{X_i, f_i\} = \{(x_i) \in \prod_{i=1}^{\infty} X_i | f_i(x_{i+1}) = x_i$ for each $i = 1, 2, \ldots\}$. For $Y = \varprojlim\{X_i, f_i\}$ and $i = 1, 2, \ldots$, a map $p_i : Y \to X_i$ is called a i-th *projection* if p_i satisfies $p_i((x_j)_{j=1}^{\infty}) = x_i$ for each $(x_j)_{j=1}^{\infty} \in Y$. It is well known that every n-dimensional continuum is an inverse limit of n-dimensional compact connected polyhedra with onto bonding maps.

3 Bing maps to Peano curves

A space is called a *Peano space* if the space is locally connected. A space X is called a *Peano curve* if X is a 1-dimensional Peano continuum. In this section, we prove the theorem of J. Song and E. D. Tymchatyn for graphs by using Levin's idea [7].

Theorem 11 (J. Song and E. D. Tymchatyn [9]) *Let X be a compactum and let Y be a Peano curve. Then the set of all Bing maps in $C(X, Y)$ is a G_δ-dense subset in $C(X, Y)$.*

Before we prove Theorem 11, we prove some lemmas. The next lemma follows from Theorem 10 which plays very important role in the proof of Lemma 13.

Lemma 12 *Let X be a compactum and let $F_1, F_2, \ldots, F_k$ $(k \geq 2)$ be pairwise disjoint closed subsets in X. Then there exist pairwise disjoint open subsets $U_1, U_2, \ldots, U_k$ such that $F_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $X \setminus \bigcup_{i=1}^{k} U_i$ is a Bing compactum.*

Proof. We will prove Lemma 12 by the induction on k. For $k = 2$, Lemma 12 holds by Theorem 10. Suppose that Lemma 12 holds for $k = 2, 3, \ldots, n-1$ ($n \geq 3$). Let $F_1, F_2, \ldots, F_n$ be pairwise disjoint closed subsets in X. By the inductive assumption there exist pairwise disjoint open subsets $U_1, U_2, \ldots, U_{n-2}, V_{n-1}$ such that $F_1 \subset U_1$, $F_2 \subset U_2$, $\ldots$, $F_{n-2} \subset U_{n-2}$, $F_{n-1} \cup F_n \subset V_{n-1}$ and $L_1 = X \setminus (\bigcup_{i=1}^{n-2} U_i \cup V_{n-1})$ is a Bing compactum. Since F_{n-1} and $(X \setminus V_{n-1}) \cup F_n$ are disjoint, there exist disjoint open subsets U_{n-1}, W_{n-1} such that $F_{n-1} \subset U_{n-1}$, $(X \setminus V_{n-1}) \cup F_n \subset W_{n-1}$ and $X \setminus (U_{n-1} \cup W_{n-1})$ is a Bing compactum. Let $U_n = W_{n-1} \setminus (X \setminus V_{n-1})$. We see that $F_i \subset U_i$ for $i = 1, 2, \ldots, n$ and $U_1, U_2, \ldots U_n$ are pairwise disjoint. And since $X \setminus (U_1 \cup U_2 \cup \cdots \cup U_n) = L_1 \cup (X \setminus (U_{n-1} \cup W_{n-1}))$ and L_1, $X \setminus (U_{n-1} \cup W_{n-1})$ are pairwise disjoint Bing compacta, $X \setminus (U_1 \cup U_2 \cup \ldots \cup U_n)$ is a Bing compactum. So $U_1, U_2, \ldots, U_n$ have the required property. This completes the proof.

The proof of the next lemma is inspired by the proof of Theorem 1. Let us recall that a compactum X is called a *graph* if X is a 1-dimensional polyhedron.

Lemma 13 *Let X be a compactum and let G be a connected graph. Then the set of all Bing maps in $C(X, G)$ is a G_δ-dense subset in $C(X, G)$.*

Proof. Let $X \subset \mathbb{I}^{\aleph_0}$ be a compactum, $f \in C(X, G)$ and $\varepsilon > 0$. Set $\mathcal{D}$ as in Definition 6 and $D_1, D_2, \ldots \in \mathcal{D}$ as in Proposition 9. Put $D_i(X, G) = \{g \in C(X, G) \mid g \text{ is a } D_i\text{-crooked map}\}$ for each $i = 1, 2, \ldots$

By Proposition 8 and 9, $\{g \in C(X, G) \mid g \text{ is a Bing map}\} = \bigcap_{i=1}^{\infty} D_i(X, G)$. By Baire theorem it is sufficient to show that $D_i(X, G)$ is an open dense subset in $C(X, G)$.

Claim 1. $D_i(X, G)$ is an open subset in $C(X, G)$. This result has been proved in [7]. For completeness, we give the proof.

Proof of Claim 1. Let $g \in D_i(X, G)$. By Proposition 7 and Since g is a closed map, we can take an open cover $\mathcal{V}$ of G such that $f^{-1}(V)$ is D_i-crooked for each $V \in \mathcal{V}$. Let δ be a Lesbegue number of the restriction of this cover to $g(X)$. Then $h \in D_i(X, G)$ for each $h \in C(X, G)$ with $d(h, g) < \delta/2$.

Claim 2. $D_i(X, G)$ is a dense subset of $C(X, G)$.

Proof of Claim 2. Let $f \in C(X, G)$ and $\varepsilon > 0$. Take a simplicial complex $\mathcal{K}$ of G such that mesh $\mathcal{K} < \varepsilon$. At first we will show that f can be approximated by a map $f' \in C(X,G)$ with the property that $f'^{-1}(p)$ is a Bing compactum for each $p \in \mathcal{K}^{(0)}$. Let $\{p_j\}_{j=1}^m = \mathcal{K}^{(0)} \setminus \{p \in \mathcal{K}^{(0)} | p$ is an endpoint of $G\}$. For $j = 1$, let $I_{1_1}, I_{1_2}, \ldots, I_{1_k} \in \mathcal{K}^{(1)}$ be all edges which contain p_1 as their endpoint, and let p_{1_ℓ} be the another endpoint of I_{1_ℓ} for $\ell = 1, 2, \ldots, k$. Take $r_\ell \in I_{1_\ell} \setminus \{p_1, p_{1_\ell}\}$ for $\ell = 1, 2, \ldots, k$. Let $A_1 = \bigcup_{\ell=1}^k I_{1_\ell}$. Since $f^{-1}([r_1, p_{1_1}]_{I_{1_1}}), f^{-1}([r_2, p_{1_2}]_{I_{1_2}}), \ldots, f^{-1}([r_k, p_{1_k}]_{I_{1_k}})$ are pairwise disjoint closed subsets in $f^{-1}(A_1)$, by Lemma 12, there exist open subsets $U_1, U_2, \ldots, U_k$ in $f^{-1}(A_1)$ such that $f^{-1}([r_\ell, p_{1_\ell}]_{I_{1_\ell}}) \subset U_\ell$ for $\ell = 1, 2, \ldots, k$, and $L_1 = f^{-1}(A_1) \setminus \bigcup_{\ell=1}^k U_\ell$ is a Bing compactum. Now, we construct $f_\ell : L_1 \cup U_\ell \to I_{1_\ell}$ such that $f_\ell | f^{-1}([r_\ell, p_{1_\ell}]_{I_{1_\ell}}) = f | f^{-1}([r_\ell, p_{1_\ell}]_{I_{1_\ell}})$ and ${f_\ell}^{-1}(p_1) = L_1$ for $\ell = 1, 2, \ldots, k$. We can take a map $f_{\ell_1} : L_1 \cup (U_\ell \setminus f^{-1}((r_\ell, p_{1_\ell}]_{I_{1_\ell}})) \to [p_1, r_\ell]_{I_{1_\ell}}$ such that $f_{\ell_1}^{-1}(p_1) = L_1$ and $f_{\ell_1}^{-1}(r_\ell) = f^{-1}(r_\ell)$. Then a map $f_\ell : L_1 \cup U_\ell \to I_{1_\ell}$ defined by $f_\ell(x) = f_{\ell_1}(x)$ if $x \in L_1 \cup (U_\ell \setminus f^{-1}((r_\ell, p_{1_\ell}]_{I_{1_\ell}})$ and $f_\ell(x) = f(x)$ if $x \in f^{-1}([r_\ell, p_{1_\ell}]_{I_{1_\ell}})$ has the required property. Define ${f'}_1 : f^{-1}(A_1) \to A_1$ by ${f'}_1(x) = f_\ell(x)$ if $x \in L_1 \cup U_\ell$ for $\ell = 1, 2, \ldots, k$. Define $f_1 : X \to G$ by $f_1(x) = f(x)$ if $x \in \mathrm{cl}(X \setminus f^{-1}(A_1))$ and $f_1(x) = {f'}_1(x)$ if $x \in f^{-1}(A_1)$. Then $d(f, f_1) < \varepsilon$ and ${f_1}^{-1}(p_1)$ is a Bing compactum.

If the step above has been done for $j \leq n-1$ $(2 \leq n \leq m)$, then do the same step for $j = n$. Then f can be approximated by a map $f_n : X \to G$ such that $f^{-1}(p_j)$ is a Bing compactum for $j = 1, 2, \ldots, n$. So we can take a map $f' : X \to G$ such that $d(f, f') < \varepsilon$ and $f'^{-1}(p_j)$ is a Bing compactum for each $j = 1, 2, \ldots, m$. And we may assume that $f'(x)$ is not an endpoint of G for each $x \in X$. So f can be approximated by a map $f' : X \to G$ such that $f'^{-1}(p)$ is a Bing compactum for each $p \in \mathcal{K}^{(0)}$. So we may assume that $A = \bigcup_{p \in \mathcal{K}^{(0)}} f^{-1}(p)$ is a Bing compactum.

Now, we will use an idea of the proof of [7, Theorem 1.8]. Let $D_i = (F_0, F_1, V_0, V_1) \in \mathcal{D}$. Take closed neighborhoods E_0, E_1 of F_0, F_1 such that $F_0 \subset E_0 \subset V_0$ and $F_1 \subset E_1 \subset V_1$. Since $D_i' = (E_0, E_1, V_0, V_1) \in \mathcal{D}$ and A is a Bing compactum, by Proposition 8 A is D'_i-crooked. By Proposition 7 there exists a neighborhood B of A such that B is D'_i-crooked. We claim that $H = B \cup \mathrm{int}E_0 \cup \mathrm{int}E_1$ is D_i-crooked.

Let $\varphi : \mathbb{I} \to H$ be a map with $\varphi(0) \in F_0$ and $\varphi(1) \in F_1$. Let $b_0 = \max\{b \in \mathbb{I} \mid \varphi(b) \in E_0\}$ and $b_1 = \min\{b \in \mathbb{I} \mid b > b_0$ and $\varphi(b) \in E_1\}$. Since B is D'_i-crooked, there exist t_0, $t_1 \in \mathbb{I}$ with $b_0 < t_0 < t_1 < b_1$ such that $\varphi(t_0) \in V_1$ and $\varphi(t_1) \in V_0$. So H is D_i-crooked.

Since $(X \setminus H) \cap F_0 = \phi = (X \setminus H) \cap F_1$, $X \setminus H$ is D_i-crooked and since $(X \setminus H) \cap A = \phi$, $A \cup (X \setminus H)$ is D_i-crooked. Let $\mathcal{K}^{(1)} = \{I_1, I_2, \ldots, I_s\}$. For each $I_j \in \mathcal{K}^{(1)}$, let p_{j_1}, p_{j_2} be the endpoints of I_j, and $X_j = f^{-1}(I_j)$ and $S_j = (X \setminus H) \cap X_j$. Define $g_j : X_j \to I_j$ such that $g_j{}^{-1}(p_{j_1}) = f^{-1}(p_{j_1}) \cup S_j$ and $g_j{}^{-1}(p_{j_2}) = f^{-1}(p_{j_2})$ for $j = 1, 2, \ldots, s$. Define $g : X \to G$ by $g(x) = g_j(x)$ if $x \in X_j$. Then, $d(f, g) < \varepsilon$ and for each $y \in G$, $g^{-1}(y) \subset H$ or $g^{-1}(y) \subset (X \setminus H) \cup A$. In both cases, $g^{-1}(y)$ is D_i-crooked. This completes the proof.

Remark 14 In the proof of Claim 1 we only use the fact that X is compact. So for each compactum X and space Y, the set of all Bing maps in $C(X, Y)$ is a G_δ-subset in $C(X, Y)$.

The following definition was given in [6].

Definition 15 Let Y be a space. We say that Y is *free* if for every compactum X the set of all Bing maps in $C(X, Y)$ is a dense subset in $C(X, Y)$.

A map $f : X \to Y$ is called an *n-dimensional map* if $\dim f^{-1}(y) \leq n$ for each $y \in Y$. Note that 0-dimensional maps are Bing maps. By the theorem of Hurewicz for mappings and dimension, we see that if X is a compactum and P is a polyhedron such that $\dim X > \dim P$, then there is no 0-dimensional map f from X to P.

We need the next lemma.

Lemma 16 *Let Y be a space. If for each $\varepsilon > 0$ there exist a free compactum Z and maps $p : Y \to Z$ and $q : Z \to Y$ such that $d(q \circ p, id_Y) < \varepsilon$ and q is a 0-dimensional map, then Y is a free space.*

Proof. Let X be a compactum and let $h : X \to Y$ be a map. By the assumption there exists a free compactum Z and maps $p : Y \to Z$ and $q : Z \to Y$ such that $d(q \circ p, id_Y) < \varepsilon$ and q is a 0-dimensional map. Since q is uniformly continuous, there exists $\delta > 0$ such that if $a, b \in Z$ satisfy $d(a, b) < \delta$, then $d(q(a), p(b)) < \varepsilon$. Since Z is free, there exists a Bing map $\varphi : X \to Z$ such that $d(p \circ h, \varphi) < \delta$. Let $\psi = q \circ \varphi$, then ψ is a Bing map because q is 0-dimensional and φ is a Bing map. And $d(h, \psi) = d(h, q \circ \varphi) \leq d(h, q \circ p \circ h) + d(q \circ p \circ h, q \circ \varphi) < \varepsilon + \varepsilon = 2\varepsilon$. So Y is a free space.

Now, we will give the proof of Theorem 11.

Proof of Theorem 11. By Remark 14, it is sufficient to show that Y is free. So we will show that Y satisfies the condition of Lemma 16. Let $h \in C(X, Y)$

and $\varepsilon > 0$. Since Y is a 1-dimensional continuum, Y can be written as $Y = \varprojlim \{G_i, f_i\}_{i=1}^{\infty}$, where G_i is a graph and $f_i : G_{i+1} \to G_i$ is surjective for $i = 1, 2, \ldots$ Since Y is Peano continuum, there exists $\varepsilon_1 > 0$ such that if $x, y \in Y$ satisfy $d(x, y) < \varepsilon_1$, then there exists an arc A in Y such that A contains x and y as its endpoints and diam$A < \varepsilon$. Let $\varepsilon_2 = \min\{\varepsilon, \varepsilon_1\}$. Take i sufficient large so that the projection $p_i : Y \to G_i$ is an ε_2-mapping. Since p_i is a closed map to a compactum, there exists $\varepsilon_3 > 0$ such that if $B \subset G_i$ satisfies diam$B < \varepsilon_3$, then diam$p_i{}^{-1}(B) < \varepsilon_2$. Let $\mathcal{K}$ be a subdivision of G_i with mesh$\mathcal{K} < \varepsilon_3$. Let $\mathcal{K}^{(0)} = \{v_j\}_{j=1}^{m}$ and $\mathcal{K}^{(1)} = \{I_\ell\}_{\ell=1}^{n}$. Take $a_j \in p^{-1}(v_j)$ for $j = 1, 2, \ldots$ Let $I_\ell \in \mathcal{K}^{(1)}$ and let v_{ℓ_1}, v_{ℓ_2} be endpoints of I_ℓ. Since diam$I_\ell < \varepsilon_3$, it follows that diam$(p_i{}^{-1}(I_\ell)) < \varepsilon_2$. Take $a_{\ell_1} \in p^{-1}(v_{\ell_1}) \cap \{a_j\}_{j=1}^{m}$ and $a_{\ell_2} \in p^{-1}(v_{\ell_2}) \cap \{a_j\}_{j=1}^{m}$. Since $d(a_{\ell_1}, a_{\ell_2}) < \varepsilon_2$, there exists an embedding $q_\ell : I_\ell \to Y$ such that diam$(q_\ell(I_\ell)) < \varepsilon$, $q_\ell(p_{\ell_1}) = a_{\ell_1}$ and $q_\ell(p_{\ell_2}) = a_{\ell_2}$. Define $q_i : G_i \to Y$ by $q_i(x) = q_\ell(x)$ if $x \in I_\ell$ for $\ell = 1, 2, \ldots, n$. Then $d(id_Y, q_i \circ p_i) < 2\varepsilon$, and $|q_i^{-1}(y)| < \infty$ for each $y \in Y$. So Y satisfies the condition of Lemma 16. This completes the proof.

Remark 17 In the proof of Lemma 13, we used an idea of M. Levin (see the proof of [7, Theorem 1.8]). Also, we can prove Lemma 13 by using an idea of J. Krasinkiewicz [6, Lemma (5.2)] (compare the proof of Lemma 13 with the proofs of Lemma 22, 23 and Theorem 24 in the next section).

4 Bing maps to polyhedra

In this section, by the method of J. Krasinkiewicz [6] we prove Theorem 24 and as an application of this theorem, we show the theorem of J. Song and E. D. Tymchatyn: the set of all Bing maps in $C(X, \mathbf{P})$ is a G_δ-dense subset in $C(X, \mathbf{P})$, where X is any compactum and $\mathbf{P}$ is any nondegenerate connected polyhedron. The next definition was given in [6].

Definition 18 Let X be a compactum and let $p \in C(X, \mathbb{I})$. We say that X is *folded relatively* p (*folded rel* p) if there exist closed subsets F_0, $F_{1/2}$, F_1 such that

(1)$F_0 \cup F_{1/2} \cup F_1 = X$.
(2)$F_0 \cap F_1 = \emptyset$.
(3)$p^{-1}(0) \subset F_0$, $p^{-1}(1) \subset F_1$.
(4)$F_0 \cap F_{1/2} \subset p^{-1}((1/2, 1])$, $F_{1/2} \cap F_1 \subset p^{-1}([0, 1/2))$.

A subset $X' \subset X$ is said to be *folded rel* p if X' is folded rel $p|X'$. A map f from X to a compactum Y is said to be *folded rel* p if $f^{-1}(y)$ is folded rel p for each $y \in Y$.

Lemma 19 (J. Krasinkiewicz [6]) *Let X be a compactum and let Y be a space. Then for each $p \in C(X, \mathbb{I})$ we have:*

(1) *If X is folded rel p, then for each $q \in C(Y, X)$ Y is folded rel $p \circ q$. In particular every subset of X is folded p.*

(2) *If F is a subset of X folded rel p, then some neighborhood of F in X is folded rel p.*

Lemma 20 (J. Krasinkiewicz [6]) *For each compactum X, there exists $\mathcal{P} = \{p_i\}_{i=1}^{\infty} \subset C(X, \mathbb{I})$ such that a closed subset $B \subset X$ is a Bing compactum if and only if B is folded rel p_i for each $i = 1, 2, \ldots$*

Lemma 21 (J. Krasinkiewicz [6]) *Let X be a compactum and let Y be a space. Then for each $p \in C(X, \mathbb{I})$, the set $\{f \in C(X, Y) | f$ is folded rel $p\}$ is an open subset in $C(X, Y)$.*

The next lemma is a key lemma in this paper. The proof is based on an idea of J. Krasinkiewicz [6, Lemma (5.2)].

Lemma 22 *Let X be a compactum and let A be a closed subset in X. Let $\varepsilon > 0$, σ^n $(n \geq 1)$ an n-dimensional simplex, and $p : X \to \mathbb{I}$ a map. If $f : A \to \partial \sigma^n$ is a Bing map and $\tilde{f} : X \to \sigma^n$ is a map with $\tilde{f}|A = f$, then there exists a map $g : X \to \sigma^n$ such that $g|A = f$, $d(\tilde{f}, g) < \varepsilon$ and g is folded rel p.*

Proof. Let $\varepsilon > 0$. Let $f : A \to \partial\sigma^n$ be a Bing map and let $\tilde{f} : X \to \sigma^n$ be a map with $\tilde{f}|A = f$. Let $\varphi : \sigma^n \times \mathbb{I} \to \sigma^n$ and $\psi : \sigma^n \times \mathbb{I} \to \mathbb{I}$ be projections. We may assume that $\tilde{f}$ satisfies $\tilde{f}^{-1}(\partial\sigma^n) = A$. Since $\tilde{f}$ is a closed map, by (2) of Lemma 19 there exists $\mathcal{V}$ which is a family of open subsets in σ^n such that $\partial\sigma^n \subset \bigcup \mathcal{V}$ and $\tilde{f}^{-1}(V)$ is folded rel p for each $V \in \mathcal{V}$. Let $r = d(\partial\sigma^n, \sigma^n \setminus \bigcup \mathcal{V})$ and let $Z = \{y \in \sigma^n | d(y, \partial\sigma^n) \geq r/2\}$. There exists $\delta \geq 0$ such that if $B \subset \sigma^n$ satisfies $\mathrm{diam} B < \delta$ and $B \cap \{y \in \sigma^n | d(y, \partial\sigma^n) = r/2\} \neq \emptyset$ then B is contained in some member of $\mathcal{V}$. Let $\mathcal{U} = \{U_i\}_{i=1}^{k}$ be a finite family of open n-discs in σ^n such that $Z \subset \bigcup \mathcal{U}$ and mesh $\mathcal{U} < \min\{\delta/2, r/2, \varepsilon\}$. We can assume that no proper subfamily of $\mathcal{U}$ covers Z. Take $a_i \in U_i \setminus \bigcup_{j \neq i} U_j$ for $i = 1, 2, \ldots$ There exist compact sets

$Z_1, Z_2, \ldots, Z_k$ such that $\bigcup_{i=1}^{k} Z_i = Z$ and $Z_i \subset U_i$ for $i = 1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$ there exists a PL n-disc D_i in U_i such that $Z_i \subset \mathrm{int} D_i$. For each $i = 1, 2, \ldots, k$, there exists an open n-disc G_i such that $D_i \cup \{a_i\} \subset G_i \subset \mathrm{cl} G_i \subset U_i$. For each $i = 1, 2, \ldots, k$, there exists a neighborhood W_i of a_i such that $W_i \subset G_i \setminus \bigcup_{j \neq i} G_j$. Let $O_1, O_2, \ldots, O_k$ be pairwise disjoint open intervals in (0,1/2). For each $i = 1, 2, \ldots, k$, take r_i, s_i, $t_i \in O_i$ such that $r_i < s_i < t_i$. For each $i = 1, 2, \ldots, k$, there exists PL $(n+1)$-disc E_i such that $(a_i, 3/4) \in \mathrm{int} E_i \subset E_i \subset G_i \times O_i \cup W_i \times (t_i, 1)$, $D_i \times [r_i, t_i] \cap E_i \subset \partial D_i \times [r_i, t_i]$ and $Q_i = D_i \times [r_i, t_i] \cup E_i$ is closed PL $(n+1)$-disc. Since $\{G_i \times O_i \cup W_i \times (t_i, 1)\}_{i=1}^{k}$ is pairwise disjoint and $Q_i \subset G_i \times O_i \cup W_i \times (t_i, 1)$ for $i = 1, 2, \ldots, k$, $Q_1, Q_2, \ldots, Q_k$ are pairwise disjoint. So there exists an isotopy $H_t : \sigma^n \times \mathbb{I} \to \sigma^n \times \mathbb{I}$ $(t \in \mathbb{I})$ such that $H_t|(\sigma^n \times \mathbb{I} \setminus \bigcup_{i=1}^{k} \mathrm{int} Q_i) = id_{\sigma^n \times \mathbb{I}}|(\sigma^n \times \mathbb{I} \setminus \bigcup_{i=1}^{k} \mathrm{int} Q_i)$, $H_t|Q_i$ is a homeomorphism of Q_i to itself and $H_1(Z_i \times \{s_i\}) \subset \psi^{-1}((1/2, 1))$ for $i = 1, 2, \ldots, k$. Let $g = \varphi \circ H_1^{-1} \circ (\tilde{f} \times p) : X \to \sigma^n$. Since $H_1|\partial\sigma^n \times \mathbb{I} = id_{\partial\sigma^n \times \mathbb{I}}$, $g|A = f$. Since $\mathrm{mesh}\mathcal{U} < \varepsilon$, $d(\tilde{f}, g) < \varepsilon$. Let $y \in \sigma^n$. Now we consider next three cases.

Case 1. If $y \in \sigma^n \setminus \bigcup \mathcal{U}$, then there exists $V \in \mathcal{V}$ such that $y \in V$. Since $g^{-1}(y) = \tilde{f}^{-1}(y) \subset \tilde{f}^{-1}(V)$, $g^{-1}(y)$ is folded rel p.

Case 2. Suppose that $y \in \bigcup \mathcal{U} \cap (\sigma^n \setminus Z)$. Let $U_1, U_2, \ldots, U_\ell$ be the all members of $\mathcal{U}$ which contain y. Let $U' = \bigcup_{i=1}^{\ell} U_i$. Since $U' \cap \{y \in \sigma^n | d(y, \partial\sigma^n) = r/2\} \neq \emptyset$ and $\mathrm{diam} U' < \delta$, there exists $V' \in \mathcal{V}$ such that $U' \subset V'$. Then $g^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_1 \circ \varphi^{-1}(y) \subset (\tilde{f} \times p)^{-1} \circ \varphi^{-1}(U') = \tilde{f}^{-1}(U') \subset \tilde{f}^{-1}(V')$. So $g^{-1}(y)$ is folded rel p.

Case 3. If $y \in Z$, there exists $i = 1, 2, \ldots, k$ such that $y \in Z_i$. Since $H_1(\{y\} \times \mathbb{I}) = H_1(\{y\} \times [0, s_i]) \cup H_1(\{y\} \times [s_i, t_i]) \cup H_1(\{y\} \times [t_i, 1])$, $H_1(\{y\} \times \mathbb{I})$ is folded rel ψ. Since $g^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_1 \circ \varphi^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_1(\{y\} \times \mathbb{I})$ and by (1) of Lemma 19 $g^{-1}(y)$ is folded rel $\psi \circ (\tilde{f} \times p) = p$.

So g is folded rel p. This completes the proof.

Lemma 23 *Let X be a compactum and let A be a closed subset in X. Let $\varepsilon > 0$, σ^n $(n \geq 1)$ an n-dimensional simplex. If $f : A \to \partial\sigma^n$ is a Bing map and $\tilde{f} : X \to \sigma^n$ is a map with $\tilde{f}|A = f$, then there exists a Bing map $g : X \to \sigma^n$ such that $d(\tilde{f}, g) < \varepsilon$ and $g|A = f$.*

Proof. Set $\mathcal{P}$ as in Lemma 20. Let $C(X, f|A) = \{g \in C(X, \sigma^n) | g|A = f\}$. For each $p_i \in \mathcal{P}$, let $C(X, f|A, p_i) = \{g \in C(X, f|A) | g$ is folded rel $p_i\}$. Let $B(X, f|A) = \{g \in C(X, f|A) | g$ is a Bing map$\}$. Since $B(X, f|A) = \bigcap_{i=1}^{\infty} C(X, f|A, p_i)$, by Lemma 21, 22 and Baire theorem $B(X, f|A)$ is dense in $C(X, f|A)$. This completes the proof.

The following theorem is a more precise result than the theorem of J. Song and E. D. Tymchatyn.

Theorem 24 (Extension Theorem of Bing Maps) *Let X be a compactum and let A be a closed subset in X. Let $\mathcal{K}$ be a finite simplicial complex such that $|\mathcal{K}|$ is a nondegenerate connected polyhedron and let $\mathcal{L}$ be a subcomplex of $\mathcal{K}$. If $f : A \to |\mathcal{L}|$ is a Bing map and $\tilde{f} : X \to |\mathcal{K}|$ is a map with $\tilde{f}|A = f$ and $\tilde{f}^{-1}(|\mathcal{L}|) = A$, then for any $\varepsilon > 0$ there exists a Bing map $g : X \to |\mathcal{K}|$ such that $g|A = f$ and $d(\tilde{f}, g) < \varepsilon$.*

Proof. First, we prove the following claim:

The set $C_{\mathcal{K}^0}(X, |\mathcal{K}|) = \{f \in C(X, |\mathcal{K}|) \mid f^{-1}(v)$ is a Bing compactum for each vertex $v \in \mathcal{K}^0\}$ is a G_δ-dense subset of $C(X, |\mathcal{K}|)$.

Let $v = v_0 \in \mathcal{K}^0$ and let $p : X \to \mathbb{I}$ be a map. We shall prove that $C_v(X, |\mathcal{K}|, p) = \{f \in C(X, |\mathcal{K}|) \mid f^{-1}(v)$ is folded rel $p\}$ is an open and dense subset of $C(X, |\mathcal{K}|)$. We can easily see that $C_v(X, |\mathcal{K}|, p)$ is an open set of $C(X, |\mathcal{K}|)$. We prove that $C_v(X, |\mathcal{K}|, p)$ is dense in $C(X, |\mathcal{K}|)$.

Let $\varepsilon > 0$ and $0 < \alpha < \beta < 1$. Consider the star $St(v, \mathcal{K}) = \cup\{\sigma \in \mathcal{K} | \; v \in \sigma\}$ of $\mathcal{K}$ with v. For each simplex $\sigma = [v_0, v_1, .., v_m] \in \mathcal{K}$ ($v_0 = v, m \geq 1$), put

$$\sigma_\alpha = \{\Sigma_{i=0}^m t_i v_i \mid t_i \geq 0 \; (i = 0, 1, 2, .., m), \; \Sigma_{i=0}^m t_i = 1, \; t_0 \geq \alpha\}$$

$$\sigma_\beta = \{\Sigma_{i=0}^m t_i v_i \mid t_i \geq 0 \; (i = 0, 1, 2, .., m), \; \Sigma_{i=0}^m t_i = 1, \; t_0 \geq \beta\}.$$

Let

$$M = \cup_{v \in \sigma \in \mathcal{K}} \sigma_\alpha, \; N = \cup_{v \in \sigma \in \mathcal{K}} \sigma_\beta.$$

Choose positive numbers s_0, s_1, and s_2 with $0 < s_0 < s_1 < 1/2 < s_2 < 1$. Consider the following set

$$Z = (M \times [s_0, s_1]) \cup (cl(M - N) \times [s_1, s_2]) \subset St(v, \mathcal{K}) \times [0, 1].$$

For each (m-dimensional) simplex σ containing v ($m \geq 1$), put $\sigma_Z = Z \cap (\sigma \times \mathbb{I})$. Also, consider the following map $\phi : Z \to T = ([0, 1-\alpha] \times [s_0, s_1]) \cup$

$([\beta-\alpha,1-\alpha]\times[s_1,s_2])\subset \mathbb{I}\times\mathbb{I}$ defined by $\phi(z)=(1-t_0,t)=(\Sigma_{i=1}^m t_i,t)$ for $z=(x,t)$, $t\in\mathbb{I}$ and $x=\Sigma_{i=0}^m t_i v_i\in[v_0,v_1,..,v_m]$. By identifying each $z\in Z$ with $\phi(z)\in T$, we obtain the adjunction space $W=(St(v,\mathcal{K})\times\mathbb{I})\cup_\phi T$. Let $q:(St(v,\mathcal{K})\times\mathbb{I})\to W$ be the natural projection.

Let D be a closed disk (=2-cell) and consider an embedding $u:\{v\}\times[s_0,s_1]\to\partial(D)$, where $\partial(D)$ is the manifold boundary of D. Also, consider the adjunction space $E=(St(v,\mathcal{K})\times\mathbb{I})\cup_u D$. Note that for each simplex σ containing v, $\mathrm{cl}(q(\sigma\times\mathbb{I})-q(\sigma_Z))$ is naturally homeomorphic to $\sigma\times\mathbb{I}$ and $q(Z)$ is homeomorphic to D. Hence W is homeomorphic to E. More precisely, for each simplex $\sigma=[v,v_1,..,v_m]\in\mathcal{K}$ containing v $(m\geq 1)$ there is an embedding $h_\sigma:q(\sigma\times\mathbb{I})\to E$ such that $h_\sigma(\mathrm{cl}(q(\sigma\times\mathbb{I})-q(\sigma_Z)))=\sigma\times\mathbb{I}$ and $h_\sigma(q(\sigma_Z))=D$ and $h_\sigma(q(H_\sigma))=\{v\}\times\mathbb{I}$, where

$$H_\sigma=\{v\}\times([0,s_0]\cup[s_1,s_2])\cup(\sigma_\alpha\times\{s_0\})\cup(H_\alpha\times[s_0,s_2])\cup$$

$$(H_\beta\times[s_1,s_2])\cup(cl(\sigma_\alpha-\sigma_\beta)\times\{s_2\})\cup(\sigma_\beta\times\{s_1\}),$$

and

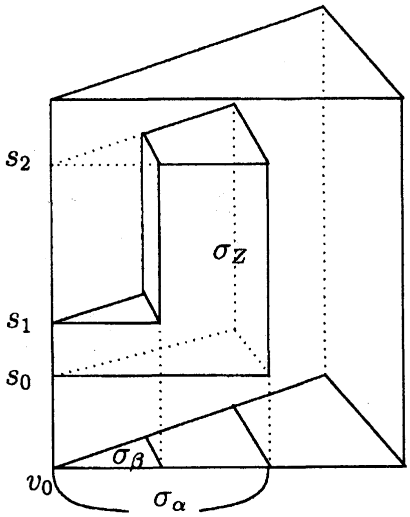
$$H_\alpha=\{\Sigma_{i=0}^m t_i v_i\mid t_i\geq 0\ (i=0,1,2,..,m),\ \Sigma_{i=0}^m t_i=1,\ t_0=\alpha\},$$

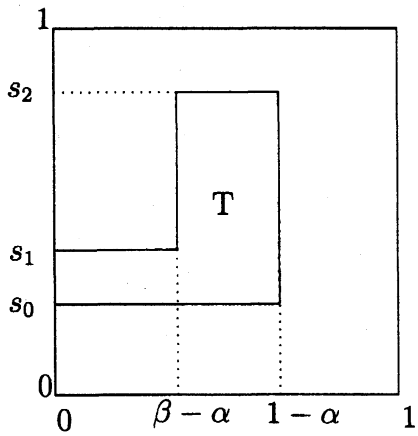
$$H_\beta=\{\Sigma_{i=0}^m t_i v_i\mid t_i\geq 0\ (i=0,1,2,..,m),\ \Sigma_{i=0}^m t_i=1,\ t_0=\beta\}.$$

Also, choose a map $u':E=(St(v,\mathcal{K})\times\mathbb{I})\cup_u D\to St(v,\mathcal{K})\times\mathbb{I}$ such that $u'|(St(v,\mathcal{K})\times\mathbb{I})=id$ and $u'^{-1}(\{v\}\times\mathbb{I})=\{v\}\times\mathbb{I}$. By using these maps, we can obtain a map $h:|\mathcal{K}|\times\mathbb{I}\to|\mathcal{K}|\times\mathbb{I}$ such that for each simplex $[v,v_1,..,v_m]\in\mathcal{K}$ containing v, $h|[v_1,..,v_m]=id$ and $h^{-1}(\{v\}\times\mathbb{I})=\cup_{v\in\sigma\in\mathcal{K}}H_\sigma$. Consider the map $g=\varphi\circ h\circ(f\times p):X\to|\mathcal{K}|$, where $\varphi:|\mathcal{K}|\times\mathbb{I}\to|\mathcal{K}|$ is the natural projection. Since H_σ is crooked with respect to p, we see that $g^{-1}(v)$ is folded rel p (see the following figures below). Since we can choose a positive number α with $1-\alpha<\epsilon$, we see that $d(f,g)<\epsilon$. Hence we see that $C_v(X,|\mathcal{K}|)=\{f\in C(X,|\mathcal{K}|)\mid f^{-1}(v)$ is a Bing compactum $\}$ is a G_δ-dense subset of $C(X,|\mathcal{K}|)$. Then

$$C_{\mathcal{K}^0}(X,|\mathcal{K}|)=\cap_{v\in\mathcal{K}^0}C_v(X,|\mathcal{K}|)$$

is a G_δ-dense subset of $C(X,|\mathcal{K}|)$. Hence the claim is true.





Let $\dim|\mathcal{K}| = n$. For each $j = 0, 1, \ldots, n$, let $A_j = |L| \cup |\mathcal{K}^{(j)}|$. By the claim, we may assume that $\tilde{f}|\tilde{f}^{-1}(A_0) : \tilde{f}^{-1}(A_0) \to A_0$ is a Bing map. Put $\tilde{g}_0 = \tilde{f}$. Note that for each simplex $\sigma \in \mathcal{K}$, the boundary $\partial\sigma$ is a Z-set of σ. By Lemma 23, we have a Bing map $g_1{:}\tilde{g}_0{}^{-1}(A_1) \to A_1$ such that $g_1|\tilde{g}_0{}^{-1}(A_0) = \tilde{g}_0|\tilde{g}_0{}^{-1}(A_0)$. By the homotopy extension theorem, we may assume that there is a map $\tilde{g}_1 : X \to |K|$ such that $\tilde{g}_1$ is an extension of g_1, and $\tilde{g}_1{}^{-1}(A_1) = \tilde{g}_0{}^{-1}(A_1)$. If we continue this process, we have a Bing map $g = g_n : X \to |\mathcal{K}|$ such that $g|A = f$ and $d(\tilde{f}, g) < \varepsilon$. This completes the proof.

The next result is the theorem of J. Song and E. D. Tymchatyn.

Corollary 25 (J. Song and E. D. Tymchatyn [9]) *Let X be a compactum and let $\mathbf{P}$ be an n-dimensional connected polyhedron ($n \geq 1$). Then the set of all Bing maps in $C(X, \mathbf{P})$ is a G_δ-dense subset in $C(X, \mathbf{P})$.*

Proof. If we put $A = |\mathcal{L}| = \phi$ in Theorem 24, we obtain this theorem.

Corollary 26 (J. Song and E. D. Tymchatyn [9]) *Let $\mathbf{M}$ be a Menger manifold with* $\dim \mathbf{M} \geq 1$. *Then the set of all Bing maps in $C(X, \mathbf{M})$ is a G_δ-dense subset in $C(X, \mathbf{M})$* (see [1] for properties of Menger manifolds).

Proof. We only prove that $\mathbf{M}$ is free. Let $\varepsilon > 0$. There exists a non-degenerate connected polyhedron $\mathbf{P} \subset \mathbf{M}$ and map $p : \mathbf{M} \to \mathbf{P}$ such that

$d(x, p(x)) < \varepsilon$ for each $x \in \mathbf{M}$ (see [1]). Let $q : \mathbf{P} \to \mathbf{M}$ be a natural embedding. Then q is 0-dimensional and $d(q \circ p, id_{\mathbf{M}}) < \varepsilon$. By Lemma 16 and Corollary 25, $\mathbf{M}$ is free. This completes the proof.

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